MATH 112: Contemporary Mathematics

Complete Lecture Notes

Spring 2026

Contents

Day 1

Problem Solving & Set Theory

January 27, 2026

Today: Problem Solving and Critical Thinking + Introduction to Set Theory.
Textbook sections 1.1-1.3, 2.1-2.2.

Warmup.

1. Below is a list of **false** statements. For each false statement, give an example that shows it is false.
   (a) Every U.S. coin is colored silver.
   (b) Every odd number ends with the letter E.
   (c) Adding 3 to any number makes its second digit larger.
2. What number comes next in each of these sequences? Explain your answer.
   (a) 1,0,0,1,0,0,1,0, _____
   (b) 2,5,8,11,14,17, _____
   (c) 1,31,531,7531, _____
   (d) 1,3,4,6,7,9,10, _____

Problem Solving and Critical Thinking

Patterns

Idea: Mathematics is all about patterns.

Example. Multiplying by 11. We have

$$\begin{aligned} 23 \times 11 &= 253 \\ 44 \times 11 &= 484 \\ 81 \times 11 &= 891 \end{aligned}$$

We make a conjecture:

To multiply a two-digit number by 11, add its digits together and put the result in between.

Now multiplication is a lot faster:

$$18 \times 11 = 198$$
$$63 \times 11 = 693$$
$$51 \times 11 = 561$$

Counterexample: $39 \times 11 = ?$ We get 3129, but it's 429.

Sometimes, you can "rescue" the pattern by changing it slightly. In this case we get

To multiply a two-digit number by 11, add its digits together and put the result in between, carrying the 1 to the first digit if necessary.

Example. 2(d) from warmup. The sequence is

$$1, 3, 4, 6, 7, 9, 10, \underline{\qquad}.$$

The differences go 2,1,2,1. So, following the pattern, we can predict that the next number is $10 + 2 = 12$.

Idea: **Sometimes there is more than one correct pattern.**

Tricks for finding patterns in lists of numbers:

- Periodic sequences
- Successive differences
- Successive factors (geom progressions)
- Skip it and come back to it

Models and Estimation

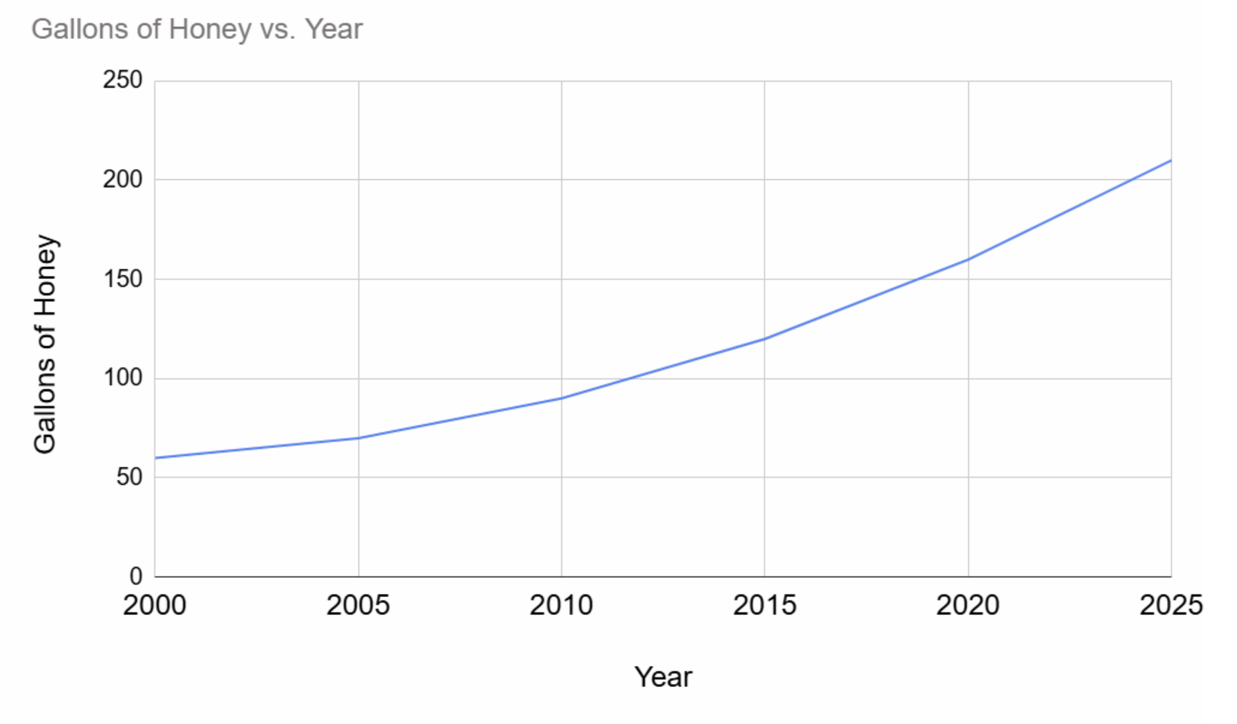
Rounding and estimation

Math uses a lot of numbers, so good to get used to working with them.
We can round big numbers. Explain how to round.

Example. Estimating using rounding.
A full-time electrician earns \$31 per hour. Estimate their weekly, monthly, and annual salary.

Estimation using charts

Example. I started a backyard honey business in 2000. Use the following line chart to estimate how many gallons of honey I sold in 2008 and 2017. How many gallons of honey should I expect to sell in 2030?



Problem Solving

Steps for solving word problems:

1. Write down the important information from the problem.
2. Make a plan for solving the problem.
3. Execute the plan.
4. Check your answer.

Example. A car costs \$3500. Given a \$1000 down payment and a \$250 monthly payment, how long will it take to pay off the car?

Part 1 Problems

1. Find a counterexample to the following false statement: Every natural number is divisible by 2,3,5, or 7.

2. Which number comes next in each sequence?
   (a) 4,19,34,49,...
   (b) 6,3,1,4,6,3,1,4,6,...
   (c) 0,1,0,0,1,0,0,0,1,0,...
3. Use rounding to estimate the number of people on Earth whose birthday is today.
4. For each of these problems, use the four problem-solving steps to write out a solution.
   (a) A movie theater sells popcorn in the following sizes: small for \$6.50, medium for \$8.00, large for \$10.50. A large popcorn is 54 ounces, a medium popcorn is 44 ounces, and a small is 16 ounces. Which of these is the best deal?
   (b) A hybrid car has an average efficiency of 40 miles per gallon and a 14.5-gallon fuel tank. Can it drive from Charlotte, NC to New York, NY on a single tank, given that the two cities are 500 miles apart?

Introduction to Set Theory

Basic Set Concepts

Sets and Elements

Definition is very simple:

Definition. Define sets and how to write them with curly braces.

Examples:

- $\{5,10,15,20\}$
- The set of all presidents of the United States
- {5,10, George Washington, 15,20}

Define elements, $\in$, $\notin$.

Three ways to write down a set: list out its elements, or we can describe the set in words. Third way is **set-builder notation**. Example:

$$\{n \mid n \text{ is an integer from 1 to 4}\}$$

Set-builder notation is really useful for when we want to write down more complicated sets. For example:

$$\{5 \times n \mid n \text{ is an integer from 1 to 40}\}.$$

Now our set has 40 elements, and it would be a real pain to write it out in full. But set-builder notation makes it easy.

Some common sets get names: examples $\mathbb{N}, \mathbb{Z}, \varnothing$. Also you can give any set a temporary name.

Cardinality and Equivalent Sets

Define cardinality, give an easy example. Mention $|\varnothing| = 0$.

Example. Infinite sets. We have $|\mathbb{N}| = \infty$. This doesn't mean infinity is a number, just that $\mathbb{N}$ is an infinite set. Define finite sets.

Cardinality lets us compare the number of elements between sets:

Definition. Define equivalent sets.

Example: $\{0, 1, \text{Mount Everest}, -1\}$ and {March, April, May, June}.

Equivalent $\neq$ equal:

Definition. Define equal sets.

Example: {March, April, May, June} and {May,March,June,April}

Subsets

Subsets and proper subsets

Example:

$$A = \{5, 10, 15, 20\}, B = \{5, 10, \text{George Washington}, 15, 20\}.$$

A is "smaller than B" in cardinality but also B has all of A's elements and more:

Definition. Define subsets, $\subseteq, \not\subseteq$.

Example. For any set, $A \subseteq A$.

Definition. Define proper subset.

Example. Explain why $\varnothing \subseteq A$ for all A.

Counting subsets

Key question: How many subsets does a set with n elements have?

Example. Let's continue with our example $A = \{5, 10, 15, 20\}$. Let's list out all the subsets of A:

$$\varnothing, \{5\}, \{10\}, \{15\}, \{20\}, \{5, 10\}\{5, 15\}, \{5, 20\}, \{10, 15\}, \{10, 20\},$$
$$\{15, 20\}, \{5, 10, 15\}, \{5, 10, 20\}, \{5, 15, 20\}, \{10, 15, 20\}, \{5, 10, 15, 20\}.$$

That's 16 subsets.

We can make a chart recording how many subsets a set with n elements has:

Cardinality	0	1	2	3	4
Number of subsets	1	2	4	8	16

The pattern is that we multiply by 2 to get from each number to the next. So a set with n elements has 2^n subsets.

Part 2 Problems

1. Write down a set that has exactly 28 elements using set-builder notation.
2. What are the cardinalities of the following sets?
   (a) $\{n \in \mathbb{N} \mid n < 6\}$
   (b) $\{n \in \mathbb{N} \mid n > 6\}$
   (c) $\{n \mid n \in \varnothing\}$
   (d) $\{n \mid n \subseteq \varnothing\}$
3. Is the empty set equal to itself? Why or why not?
4. Write down three sets A, B, C such that A is a proper subset of B and B is a proper subset of C.
5. For each of the finite sets in problem 2, find how many subsets the set has.

Day 2

Set Theory & Venn Diagrams

February 3, 2026

Today: More on set theory.
Textbook sections 2.1-2.5.

Introduction to Set Theory

Basic Set Concepts

Sets and Elements

Recall:

- three ways to write sets, examples for each.
- Sets that get special names: $\mathbb{N}, \mathbb{Z}, \varnothing$.
- Definition of cardinality.

Mention finite vs. infinite sets (said a little last time, but worth repeating)

Equal and Equivalent Sets

Definition. Define equivalent sets.

Example: $\{0, 1, \text{Mount Everest}, -1\}$ and {March, April, May, June}.

Equivalent $\neq$ equal:

Definition. Define equal sets.

Example: {March, April, May, June} and {May,March,June,April}

Subsets

Subsets and proper subsets

Example:

$$A = \{5, 10, 15, 20\}, B = \{5, 10, \text{George Washington}, 15, 20\}.$$

A is "smaller than B" in cardinality but also B has all of A's elements and more:

Definition. Define subsets, $\subseteq, \not\subseteq$.

Example. For any set, $A \subseteq A$.

Definition. Define proper subset.

Example. Explain why $\varnothing \subseteq A$ for all A.

Counting subsets

Key question: **How many subsets does a set with n elements have?**

Example. Let's continue with our example $A = \{5, 10, 15, 20\}$. Let's list out all the subsets of A:

$$\varnothing, \{5\}, \{10\}, \{15\}, \{20\}, \{5, 10\}\{5, 15\}, \{5, 20\}, \{10, 15\}, \{10, 20\},$$
$$\{15, 20\}, \{5, 10, 15\}, \{5, 10, 20\}, \{5, 15, 20\}, \{10, 15, 20\}, \{5, 10, 15, 20\}.$$

That's 16 subsets.

We can make a chart recording how many subsets a set with n elements has:

Cardinality	0	1	2	3	4
Number of subsets	1	2	4	8	16

The pattern is that we multiply by 2 to get from each number to the next. So a set with n elements has 2^n subsets.

Part 1 Problems

1. Is $\mathbb{N} \subseteq \mathbb{Z}$? Is $\mathbb{Z} \subseteq \mathbb{N}$?
2. How many proper subsets does the set $\{3, 0, 1\}$ have?
3. Write down three sets A, B, C such that A is a proper subset of B and B is a proper subset of C.

More Set Theory

Venn Diagrams and Set Operations

Define Venn diagrams, draw some. We can represent different relationships between sets with Venn diagrams:

- disjoint sets;
- proper subsets;

- equal sets;
- sets with some elements in common.

Definition. Universal set, complement, intersection, union.

Complements, intersections, unions involving the empty set.

More Venn diagrams: performing set operations based on diagrams.

Mathematical meaning of *or* and *and*.

Cardinality of unions: $|A \cup B| = |A| + |B| - |A \cap B|$.

Operations and Diagrams with Three Sets

Example. Let the universal set be $U = \{2, 4, 6, 8, 10, 12\}$ and set

$$\begin{aligned} A &= \{2, 6, 8\}, \\ B &= \{2, 4, 8\}, \\ C &= \{2, 10\}. \end{aligned}$$

Find $A \cap (B \cup C)$ and $(A \cup B) \cap (A \cup C)$.

Venn diagram with three sets. Determining a Venn diagram from given sets.

Example. Construct a Venn diagram illustrating A, B, C above.

Survey Problems

Example. A survey asked 1800 Americans whether they carry a credit card, a debit card, both, or neither. 1020 people said they carry a credit card, and 445 said they carry both, while 120 people carry neither. Draw a Venn diagram to illustrate these results. How many people carry only a debit card?

Example. A group of 80 people was asked if they owned dogs, cats, or any other pets. The results were:

- 42 said they had dogs,
- 34 said they had cats,
- 27 said they had another pet,
- 12 said they had dogs and another pet,
- 14 said they had dogs and cats,
- 10 said they had cats and another pet, and
- 7 said they had all three.

Draw a Venn diagram to illustrate this, and answer the following.

1. How many had only dogs?
2. How many had cats and another pet, but no dogs?
3. How many had exactly one kind of pet?
4. How many had at least two kinds of pets?
5. How many had no pets at all?

Part 2 Problems

1. What is the intersection of the even numbers and the odd numbers?
2. Suppose $A \subseteq B$. What is $A \cup B$? What is $A \cap B$?
3. How many single digit numbers contain the letter e or the letter i? How many contain the letter e and the letter i? Draw a Venn diagram to represent this.
4. A survey of 380 people found that 95 eat fast food at least twice a week, while 160 never eat fast food. How many people eat fast food at least once per week? How many people eat fast food exactly once per week?

Day 3

Logic I: Statements, Connectives & Truth Tables

February 10, 2026

Today: Introduction to logic.
Textbook sections 3.1-3.4.

Statements, Negations, and Quantified Statements

Statements

Definition. A *statement* is a sentence that is either true or false.

Just like sets, we can give symbolic names to statements. Give some examples.

Definition. If p is a statement, the *negation* of p is the statement expressing that p is false. We denote it by $\sim p$.

Examples of negations. (Avoid quantified statements!)

> **Important for later:** for any statement p, one of p and $\sim p$ is true and the other is false.You can use this to check if you got the negation right.

We can go between symbolic names for statements and their word form:

Example.

1. Let p denote the statement "Texas is a country." Write $\sim p$ in words.
2. Write down a statement q such that $\sim q$ is the statement "Today is not Monday."

The book skips this, but maybe mention that $\sim\sim p$ and p mean the same thing.

Quantified statements

Definition. A *quantified statement* is a statement that uses "all", "every", "some", or "no".

Give lots of examples.

Walk through negating a quantified statement. What is the negation of the following statements?

- Every cow has spots.
- Some people do not recycle.
- No trees have blue leaves.

Negating quantified statements:
If p is "All X are Y", then $\sim p$ is "Some X are not Y" or "Not all X are Y".
If p is "Some X are Y", then $\sim p$ is "No X are Y" or "All X are not Y".
If p is "No X are Y", then $\sim p$ is "Some X are Y".

Connectives

Definition. A *connective* is a symbol that lets us connect statements together to make new *compound* statements.

Some connectives: *and, or.* Introduce the notation $p \wedge q, p \vee q$ via examples.

Example. Let p be "It is raining outside" and let q be "It was snowing outside yesterday." Write the following out in words: (i) $p \vee \sim q$, (ii) $\sim(p \wedge q)$.

Two new connectives: *implication* $\rightarrow$, *biconditional* $\leftrightarrow$. Introduce with easy examples. Explain the wording "if and only if".

Example. Consider the statement "If today is not Sunday and the time is earlier than 6pm, then the store is open." Write this statement symbolically using connectives.

Part 1 Problems

1. For each of the following statements, write down its negation in words.
   (a) Everyone who has an allergy is born with it.
   (b) France is the capital of Paris.
   (c) Nobody likes both rock and classical music.
   (d) Every animal is a mammal or a reptile.

2. Let p be "Today is Saturday" and let q be "Mr. Schachner slept in this morning". Express the following statements in words.
   (a) $p \wedge q$
   (b) $p \leftrightarrow \sim q$
   (c) $\sim p \to q$
3. For the same p and q as the previous problem, express the following statements symbolically.
   (a) Today is not Saturday, or Mr. Schachner slept in this morning.
   (b) Today is Saturday if and only if Mr. Schachner slept in this morning.
   (c) It is not both true that today is Saturday and that Mr. Schachner did not sleep in this morning.

Truth tables

Some vocab: the *truth value* of a statement is just whether it is true or false. Statements can have one of two truth values: true or false.

Definition. A *truth table* for a compound statement is a chart that describes the truth value of the statement given the truth values of its parts.

Example. Let p be "It is cold outside" and let q be "The sun is up". Write down truth tables for $p \wedge q$ and $p \vee q$.

p	q	$p \wedge q$	$p \vee q$

In fact, the truth table for a compound statement like $p \vee q$ doesn't depend on what p and q actually are– just whether they are true or false. So we can write down truth tables for $p \wedge q$ and $p \vee q$ for any p and q. Write these out on the board.

We can also write down truth tables for more complicated statements:

Example. Write down truth tables for the following statements: (i) $\sim(\sim p \vee q)$, (ii) $p \wedge \sim p$, (iii) $p \vee (q \wedge r)$.

Truth tables for the conditional and biconditional

Spend some time talking about the truth value of $p \to q$ when for each of the possible truth values of p and q. The hard case is when p is false: the book frames this as $p \to q$ being a promise, which can only be broken if p is true and q is false.

Write down the full truth tables for implication and biconditional.

Example. Write down truth tables for the following statements: (i) $p \to \sim q$, (ii) $p \vee (q \leftrightarrow r)$, (iii) $(p \to q) \vee (q \to p)$.

Part 2 Problems

1. Consider the statement "If it's dark outside, I don't wear sunglasses." What is the truth value of this statement if I am wearing sunglasses and it's bright outside?
2. Write down two statements p and q such that $p \leftrightarrow q$ is true, but $p \vee q$ is false.
3. For each of the following statements, write down a truth table. You can combine them all into a single truth table or write down separate tables for each.
   (a) $p \wedge q \wedge r$
   (b) $\sim(p \vee q \vee r)$
   (c) $p \leftrightarrow (q \leftrightarrow r)$

Day 4

Logic II: Equivalence, Conditionals & de Morgan

February 24, 2026

Today: Logic, part 2.
Textbook sections 3.5-3.6.

Review from last time

- **Statements** are true or false.
- The **negation** of p, written $\sim p$, is true when p is false.
- **Quantified statements** use words like "**all**", "**some**", "**every**", or "**no**".

> **Negating quantified statements:**
> If p is "All X are Y", then $\sim p$ is "Some X are not Y" or "Not all X are Y".
> If p is "Some X are Y", then $\sim p$ is "No X are Y" or "All X are not Y".
> If p is "No X are Y", then $\sim p$ is "Some X are Y".

- **Connectives** combine statements to make new statements:

$$p \wedge q \text{ means "} p \text{ and } q\text{"}$$
$$p \vee q \text{ means "} p \text{ or } q\text{"}$$
$$p \to q \text{ means "if } p \text{ then } q\text{"}$$
$$p \leftrightarrow q \text{ means "} p \text{ if and only if } q\text{"}$$

- We can form **truth tables** for a statement by writing down all possible values of the variables and plugging them in and simplifying:

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Equivalent Statements

Example. Show that $p \to q$ and $\sim p \vee q$ have the same truth table.

p	q	$p \to q$	$\sim p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

We can assign meanings to p and q. For example, if p is "It's cold outside" and q is "The game is canceled", then $p \to q$ is "If it's cold outside, then the game is canceled" and $\sim p \vee q$ is "It's not cold outside or the game is canceled." These statements are **equivalent:**

Definition. Two statements are **equivalent** if they have the same truth table.

Example. Consider the statement "If the movie's longer than 2 hours, I'm gonna fall asleep." Which of the below statements is equivalent to this statement?

- If the movie's shorter than 2 hours, then I'm not gonna fall asleep.
- If I'm gonna fall asleep, then the movie's longer than 2 hours.
- If I'm not gonna fall asleep, then the movie is shorter than 2 hours.

Let p be "The movie's longer than 2 hours", and let q be "I'm gonna fall asleep". We want to find a statement equivalent to $p \to q$. The statements are

- $\sim p \to \sim q$,
- $q \to p$,
- $\sim q \to \sim p$,

The truth table is

p	q	$p \to q$	$\sim p \to \sim q$	$q \to p$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

So the third statement is equivalent to $p \to q$.

Variations of Conditional Statements

As we just saw, the statement $p \to q$ is not equivalent to $q \to p$ or $\sim p \to \sim q$. But it is equivalent to $\sim q \to \sim p$. These variations have special names:

- The **inverse** of $p \to q$ is $\sim p \to \sim q$.
- The **converse** of $p \to q$ is $q \to p$.
- The **contrapositive** of $p \to q$ is $\sim q \to \sim p$.

Every conditional statement is not equivalent to its inverse or converse, but it is always equivalent to its contrapositive.

Example. Consider the conditional "If you're going, I'm going." Write down the inverse, converse, and contrapositive, and indicate which of these is equivalent to the original statement.

Inverse: If you're not going, I'm not going.
Converse: If I'm going, you're going.
Contrapositive: If I'm not going, you're not going. **Equivalent.**

Part 1 Problems

1. All of the following statements are equivalent to each other, except for one. Which one?
   (a) If you haven't been to New York, you haven't been to America.
   (b) You've been to America or you haven't been to New York.
   (c) If you've been to New York, you've been to America.
   (d) If you haven't been to America, you haven't been to New York.
2. Can a statement be equivalent to its own negation? Explain your reasoning.
3. Can a conditional statement be equivalent to its converse?
4. For each of the following statements, express it symbolically, then write down the inverse, converse, and contrapositive.
   (a) If you have a question, you raise your hand.
   (b) If the bus is late, I will miss class.

Negations of Conditional Statements

What is the negation of $p \to q$? Remember that $p \to q$ can only be false if p is true and q is false. This means that $\sim(p \to q)$ can only be **true** if p is true and q is false. So

$$\sim(p \to q) = p \wedge \sim q.$$

Example. Determine the negation of the following statements.

- If you're not first, you're last. You're not first and you're not last.
- If you learned logic, you're better at winning arguments. You learned logic and you're not better at winning arguments.
- If there's no free food, I'm not coming. There's no free food and I'm coming.

When asked to find the negation of a compound symbolic statement, you want to express its negation such that the symbol $\sim$ negates only simple statements.

de Morgan's Laws

What about negations of statements built using connectives?

Example. Let p be "My sandwich has peanut butter" and let q be "My sandwich has jelly". What is $\sim(p \wedge q)$? What is $\sim(p \vee q)$?

These rules hold in general:

de Morgan's Laws
$\sim(p \wedge q)$ is equivalent to $\sim p \vee \sim q$.
$\sim(p \vee q)$ is equivalent to $\sim p \wedge \sim q$.

Example. Use de Morgan's laws to find a statement equivalent to the negation of "We have questions and you have answers."

Part 2 Problems

1. Determine the negations of the following statements.
   (a) If it's too dark to see, I'll bring my flashlight.
   (b) If the car has run out of gas, it stops.
   (c) If you use a lot of electricity, your bill will be expensive.
   (d) If my dog is happy, he wags his tail.

2. Use de Morgan's laws to write a statement that is equivalent to the negation of each statement.
   (a) I didn't sleep well and I'm tired.
   (b) The plant is being watered too much, or it's not being watered enough.
   (c) She brushed her teeth and washed her face.
3. The statement $\sim(p \to (p \vee q))$ is always false, no matter what p and q are. Show this fact in two ways: first using a truth table, then using negations of conditionals and de Morgan's laws.

Day 6

Logic III & Introduction to Number Representation

February 24, 2026

Today: Logic, part 3; Intro to number representation.
Textbook sections 3.7-3.8, 4.1.

Arguments

Arguments using conditionals

Consider the following argument:

> **Argument 1.** I read on the grocery store's website that they close if there's more than two feet of snow outside. There's only six inches of snow outside today. So the grocery store is definitely open.

Explain what's wrong, puzzle it out. Next:

> **Argument 2.** Whenever I put too much lemon juice in my lemonade, my daughter complains it's too sour. She didn't complain when I made her lemonade after school today, so I didn't use too much lemon juice.

Explain what's right. Talk about nuances re: possible loopholes in reasoning.

Arguments using quantified statements

Three arguments. Valid or invalid?

> **Argument 3.** Everyone who has a life-threatening allergy carries an Epi-Pen in case of a reaction. I saw my aunt carrying an Epi-Pen in her purse. So she has a life-threatening allergy.

> **Argument 4.** A film critic on TV angrily declared that movie theaters never sell out anymore. But I saw a movie in theaters last week, and the theater was completely full. So the critic must have been wrong.

> **Argument 5.** I saw a newspaper headline saying that some gas stations are charging more than $5 for a gallon of gas. But I haven't seen gas prices above $4 since 2022. So the newspaper story must be false.

Arguments using connectives

Three arguments. Valid or invalid?

Argument 6. My doctor told me I needed to start taking vitamin D supplements. The label on a bottle of multivitamins reads "contains recommended daily intake of vitamins C, D, E, and B-12". So I can fulfill my doctor's requirement by taking the multivitamins.

Argument 7. At work we get paid on either Thursday or Friday. Today's Friday, and we didn't get paid yesterday, so we'll get paid today.

Argument 8. The Clippers are playing the Jazz next week. I don't like losing bets, so I bet one coworker that the Clippers would score 100 points, and another coworker that the Jazz would score 100 points. That way I win at least one of the bets.

Combination arguments

Argument 9. A team of biologists wanted to test the claim that *every species of mammal that has more than 30 teeth is either a carnivore* (eats only meat) *or an omnivore* (eats both meat and plants). After weeks of study, one field researcher finds a species of mammal which eats only plants and has 32 teeth. She thinks this disproves the claim, but another researcher disputes this, since the claim is only about omnivores and carnivores. Who is correct?

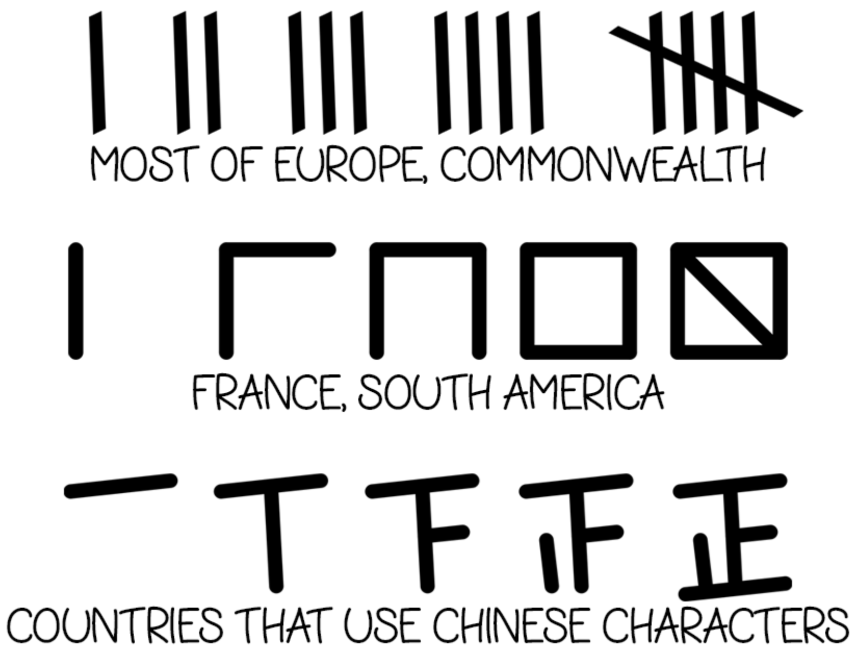
Argument 10. Changing a password online is so annoying. One website told me the following: *your password must contain both an uppercase letter and a number, or else be at least fourteen characters long.* I tried to change my password to "schachner12345" and it wouldn't accept it as a password. So the website must be broken!

Number representation

In this unit we'll think about how we write numbers down. Before we dig into our positional number system, today we'll explore ways different cultures have written numbers down.

Tally marks

Talk about different ways to write tally marks.



Roman numerals

Individual symbols have meanings:

1	5	10	50	100	500	1000
I	**V**	**X**	**L**	**C**	**D**	**M**

The numbers from 1 to 20 are:

1	2	3	4	5	6	7	8	9	10
I	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**IX**	**X**
11	12	13	14	15	16	17	18	19	20
XI	**XII**	**XIII**	**XIV**	**XV**	**XVI**	**XVII**	**XVIII**	**XIX**	**XX**

Writing a larger symbol and then a smaller symbol means *add* the values. Writing a smaller symbol and then a larger symbol means *subtract* the values.

Example.

$$\begin{aligned}
\mathbf{VII} &= 5+1+1 = 7\\
\mathbf{XIV} &= 10+5-1 = 14\\
\mathbf{XXXIX} &= 10+10+10-1 = 29\\
\mathbf{XL} &= 50-10 = 40\\
\mathbf{XC} &= 100-10 = 90\\
\mathbf{MMXXVI} &= 1000+1000+10+10+5+1 = 2026
\end{aligned}$$

Babylonian numerals

Only two symbols: $\vee = 1, <= 10$.

For numbers up to 60, things go as expected:

$$\begin{aligned}
<< \vee \vee \vee &= 23\\
\vee \vee \vee\vee &= 4\\
<<< \vee \vee \vee \vee \vee &= 35
\end{aligned}$$

Once we get to 60, things "roll over" like a clock:

$$\begin{aligned}
\vee \quad < \vee\vee &= \text{“1 minute, 12 seconds”} = 72\\
\vee \vee \vee \quad <<<< \vee \vee \vee \vee \vee\vee &= \text{“3 minutes, 46 seconds”} = 226\\
<< \vee \vee \vee \vee \vee \vee \vee \vee \quad \vee &= \text{“28 minutes, 1 second”} = 1681
\end{aligned}$$

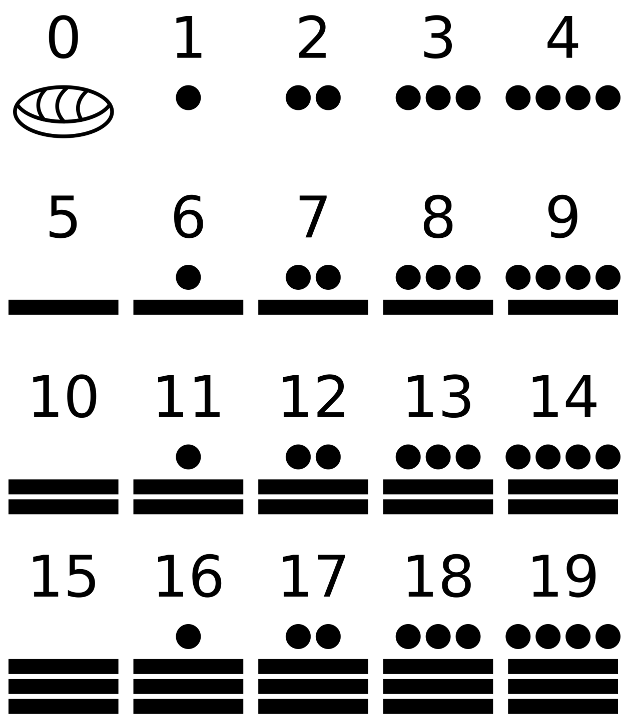
And we can roll over again once the "minutes" get to 60:

$$< \vee \quad << \quad < \vee \vee \vee = \text{“1 hour, 20 minutes, 13 seconds”} = 4813$$

Major problem: There's no way to write 0. How do we represent 120, for example?

Mayan numerals

This time, the "roll over" number is 20. And there's a symbol for 0.



Day 7

Number Bases & Binary

March 3, 2026

Today: Number representation, day 2.
Textbook sections 4.2, 4.3.

Review from last time

Last week we saw alternative numeral systems:

- Tally marks,
- Roman numerals,
- Babylonian numerals (like time: rollover number 60),
- Mayan numerals (rollover number 20).

Bases in Positional Systems

Our system

We use a system with a "roll over number" of 10, which has 10 different digits:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9.$$

When we write a number like 2, we really mean

$$349 = 3 \times 100 + 4 \times 10 + 2 \times 1.$$

Or, using exponents,

$$342 = 3 \times 10^2 + 4 \times 10^1 + 2 \times 1.$$

We say that 342 has a 3 in the hundreds place, a 4 in the tens place, and a 2 in the ones place.

So everything revolves around the number 10. Why? 10 fingers!

Writing down numbers on Mars

Fun fact: Martians are real and they have 4 fingers on each hand, for a total of 8 fingers. So they use a system with a rollover number of 8, which has 8 different digits:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7.$$

When a Martian counts, it goes

$$0, 1, 2, 3, 4, 5, 6, 7, \mathbf{10,11,12,13,14,15,16,17,20,21,22,} \ldots$$

Eventually they get to $\ldots 74, 75, 76, 77$, then the next number is 100.

Of course, they have the **same numbers**: they just write them down differently. We write

$$(10)_8 = 8, \quad (11)_8 = 9, \quad (12)_8 = 10, \quad \text{etc.}$$

In the same way that our number system revolves around 10, the Martian number system revolves around the number 8:

$$(342)_8 = 3 \times 8^2 + 4 \times 8^1 + 2 \times 1 = 169.$$

Since $8^2 = 64$, instead of a hundreds, tens, and ones place we have a 64s, 8s, and 1s place.

Example. Let's write Martian numbers as Earth numbers:

$$\begin{aligned} (31)_8 &= 3 \times 8 + 1 = 25 \\ (117)_8 &= 1 \times 8^2 + 1 * 8 + 7 = 79 \\ (10003)_8 &= 1 \times 8^4 + 3 = 4099. \end{aligned}$$

Definition. The **base** of a number representation system is the "rollover number" of that system.

In a base-b system, there are b different digits:

$$0 \quad 1 \quad 2 \quad \cdots \quad b-2 \quad b-1.$$

The **place values** are 1, b, b^2, b^3, etc.
And everything revolves around the number b. Our system is **base 10**, aka **decimal**.

Note: If b is bigger than 10 then we typically use letters to represent the extra digits. For example, a common base used by computers is base 16, where the digits are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.$$

Here A through F represent the numbers 10 through 15. Let's do a conversion:

$$(A3C)_{16} = 10 \times 16^2 + 3 \times 16 + 12 = 2620.$$

Binary

You may have heard that computers only think in 1's and 0's. What we mean by that is that computers use a base 2, or **binary** number system. Here the only digits are 0 and 1. So, a computer counts like

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, \ldots$$

Let's try some conversions:

$$\begin{aligned}(11)_2 &= 1 \times 2 + 1 = 3,\\ (1010)_2 &= 1 \times 2^3 + 1 \times 2 = 10,\\ (11111)_2 &= 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 1 = 31.\end{aligned}$$

Here, the place values are the powers of 2:

$$1, 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, \ldots$$

We can use this to write down any number in binary:

- Find the largest power of two which is less than or equal to your number.
- Put a 1 in that place.
- Subtract the power of two you found from the number. This is your new number.
- Repeat until your number runs out.

Example. Let's write 13 in binary. Following the steps:

- The largest power of 2 which is less than or equal to 140 is $2^3 = 8$. So we put a 1 in the 8s place.
- Subtracting off 8 from 13 gives 5. Now the biggest power of 2 is 4. We put a 1 in the 4s place.
- Subtracting off 4 from 5 gives 1. The largest power is now 1 itself, so we put a 1 in the 1s place.
- We subtract off 1 and get to 0. We've run out, so we're done!

We get $13 = (1101)_2$.

Example. Let's write 140 in binary. Following the steps:

- The largest power of 2 which is less than or equal to 140 is $2^7 = 128$. So we put a 1 in the 128s place.
- Subtracting off 128 from 140 gives 12. Now the biggest power of 2 is 8. We put a 1 in the 8s place.
- Subtracting off 8 from 12 gives 4. The largest power is now 4 itself, so we put a 1 in the 4s place.
- We subtract off 4 and get to 0. We've run out, so we're done!

We get $140 = (10001100)_2$.

Part 1 Problems

1. What is the base of the Mayan number system? What are its place values?
2. Is 208 a number on Mars? Why or why not?
3. What number does 31 represent on Mars? What does it represent in base 16?
4. Convert the following numbers to binary:
   (a) 2
   (b) 17
   (c) 25
   (d) 200
   (e) 111

Converting base 10 to other bases

We already saw how to write a base 10 number in binary. Here's how to write a base 10 number in base b:

- Divide your number by b, circle the remainder. The quotient becomes the new number.
- Divide that number by b again, circle the remainder. The next quotient becomes the new number again.
- Repeat until you get a number less than b. Circle that number.
- The original number in base b is now given by your circled numbers in reverse.

Example. Let's do the following base conversions:

(a) 25 to base 5 100

(b) 81 to base 3 10000

(c) 74 to base 8 112

(d) 156 to base 7 312

(e) 219 to base 16. DB

Doing arithmetic in other bases

Done exactly the same as in base 10, but the rollover number is different!

Example. Let's do the following arithmetic problems:

(a) $(12)_5 + (13)_5$ 30

(b) $(44)_6 + (55)_6$ 143

(c) $(62)_8 - (35)_8$ 27

(d) $(23)_4 \times (3)_4$ 201

(e) $(3B)_{16} + (4E)_{16}$ 89

Part 2 Problems

1. Do the following base conversions:

   (a) 16 to base 4 100

   (b) 64 to base 8 100

   (c) 89 to base 6 225

   (d) 247 to base 9 304

   (e) 763 to base 16. 2FB

2. Do the following arithmetic problems:

   (a) $(21)_3 + (12)_3$ 110

   (b) $(55)_7 - (26)_7$ 26

   (c) $(34)_6 \times (4)_6$ 224

   (d) $(276)_8 - (143)_8$ 133

   (e) $(2A)_{16} \times (3)_{16}$ 7E

Day 8

Geometry I: Angles, Polygons & Area

March 10, 2026

Today: Geometry, day 1.
Textbook sections 10.1, 10.3, 10.4, 10.5.

Points, lines, planes, & angles

Define: **Point, line, line segment, ray, plane.**

Talk about **angles**: a **vertex** formed by two lines, line segments, or rays which intersect.

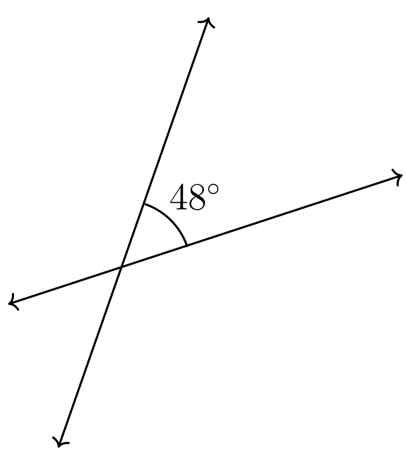
We can measure angles using **degrees**. Draw pictures of angles of all kinds. Define **Acute, right, obtuse, straight** angles.

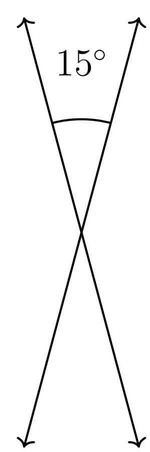
Define **complementary** and **supplementary** angles.

Vertical angles and parallel lines

Define **vertical angles**. Key idea: **Vertical angles are equal.**

Example. Find all the angles in the following diagrams.

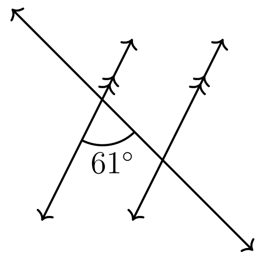


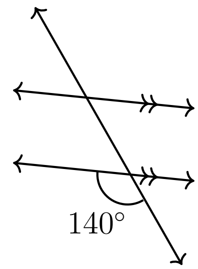


So if we know the measure of one of the angles at an intersection, we can figure out all four!

Define **parallel lines** and **corresponding angles**. Key idea: **Corresponding angles are equal.**

Example. Find all the angles in the following diagrams.



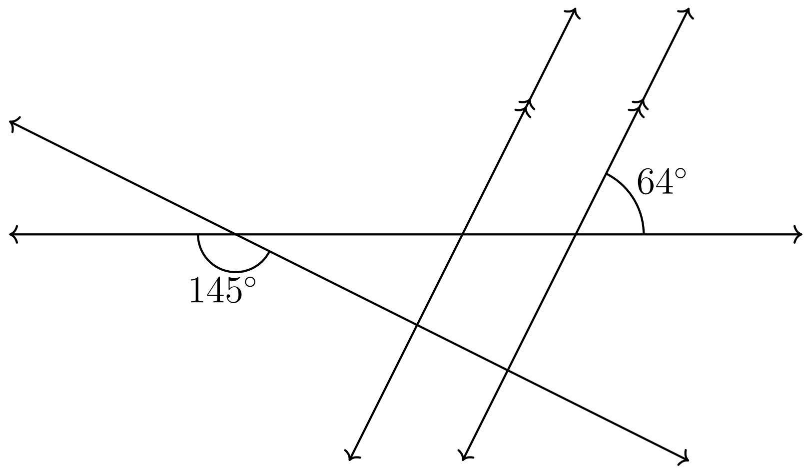


We can use this to show a cool fact: the sum of the angles of any triangle is 180°!

Part 1 Problems

1. For each of the following angle measures, determine if the angle is acute, right, obtuse, or straight. Then draw a picture of the angle.
   (a) 44°
   (b) 100°
   (c) 46°
   (d) 180°
2. Which of the previous angles are complementary to each other? Which are supplementary?
3. Can an angle be complementary to itself? Can an angle be supplementary to itself?
4. Angle A is complementary to angle B and angle C is complementary to angle A. Explain why angle B and angle C are complementary, both in words and using a picture.

5. Determine all the angles in the following diagrams.



Polygons

Define:

- **polygon**
- **triangle, quadrilateral, pentagon, hexagon, heptagon, octagon**
- Types of quadrilaterals: **parallelogram, rhombus, rectangle, square, trapezoid**.

We saw earlier that the angles in a triangle always add to 180°. What about other shapes? Show how to split up into triangles and add up to get

$$\text{Sum of angles} = 180^\circ \times (\text{number of sides} - 2).$$

Perimeter, area, circumference

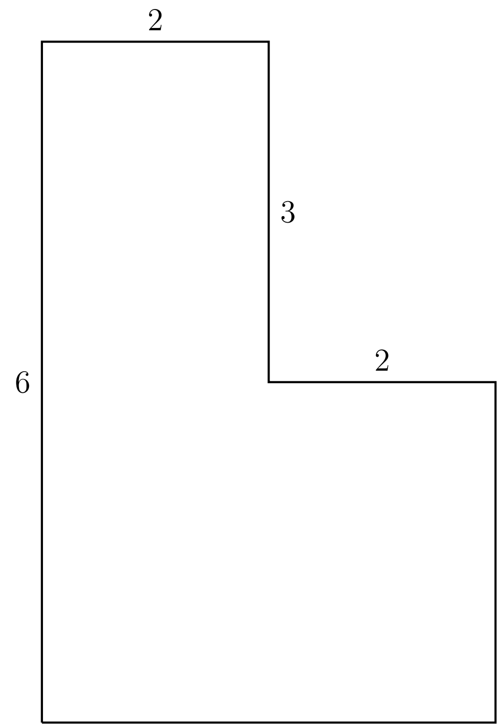
Define perimeter.

Example. A rectangular plot measures 36 yards by 48 yards. Fencing costs \$6 per foot. How much will it cost to fence the entire plot?

Define area. Write area formulas for rectangles and squares.

Example. I need to decide between two canvases. Both are 25cm wide, but one is square and the other is 40cm long. What is the area of each canvas?

Example. What is the area of the following shape?



Give other area formulas:

- **Triangle:** Area $= \frac{1}{2} \times \text{base} \times \text{height}$.
- **Parallelogram:** Area $= \text{base} \times \text{height}$.
- **Trapezoid:** Area $= \frac{1}{2} \times (\text{base}_1 + \text{base}_2) \times \text{height}$.

Example. Mr. Schachner is trying to decide between a few oddly-shaped rugs. They all have the same height of 8 feet, but one is a trapezoid with bases 4 feet and 7 feet; another is a triangle with base 9 feet; and another is a parallelogram with base 5 feet. Which rug is the biggest? Which is the smallest?

Circles: define **diameter, radius, circumference**.

Define the number π: The circumference of a circle divided by its diameter. Always the same for any circle!

Formulas:

$$\text{Circumference} = 2 \times \pi \times \text{radius}, \qquad \text{Area} = \pi \times \text{radius}^2.$$

Example. Which has more pizza, one 14-inch diameter pizza or *three* 8-inch diameter pizzas? Which has more crust?

3D solids: volume and surface area

Student version has the following table:

Shape	Volume	Surface Area
Rectangular solid	$V = lwh$	$S = 2(lw + lh + wh)$
Cube	$V = s^3$	$S = 6s^2$
Pyramid	$V = \frac{1}{3}Bh$	–
Cone	$V = \frac{1}{3}\pi r^2 h$	–
Cylinder	$V = \pi r^2 h$	$S = 2\pi r^2 + 2\pi r h$
Sphere	$V = \frac{4}{3}\pi r^3$	–

(Blanks are formulas not in the textbook)

Define **solid**. Examples: **rectangular solid, cube, pyramid, cone, cylinder, sphere**.

Write volume formulas for rectangular solid and cube.

Example. UPS charges \$15 per cubic foot to ship boxes in bulk. I managed to stuff all my belongings into three boxes, each 3 by 3 by 4 feet, except for my valuables: those went in a little cube box 2 feet on a side. How much will it cost to ship everything?

Volume formulas for pyramid, cone, cylinder, sphere.

Example. The Great Pyramid at Giza is about 500 feet tall and 750 feet on each side. If the Pharaoh had demanded a cone-shaped "pyramid" with a radius of 750 feet and the same height, how much more stone would have been needed?

Define surface area. Write formulas for surface area of rectangular solid, cube, cylinder.

Example. Which has more frosting, a two-layer cake with diameter 10 inches, or a seven-layer cake with diameter 5 inches? Assume layers are 1 inch thick each, and we frost the top but not the inner layers.

Day 9

Geometry II: Triangles, Pythagoras & Trigonometry

March 17, 2026

Today: Geometry, day 2.
Textbook sections 10.2, 10.6, 10.7.

Triangles

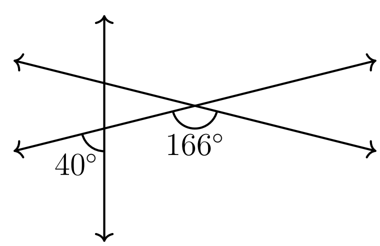
Draw some triangles! We can classify them by their sides or by their angles.

By sides: **equilateral, isoceles, scalene**.
By angles: **acute, right, obtuse**.
No matter the triangle, **The sum of the angles is always 180°.**

We can use this fact to solve problems:

Example. A triangle has one 10° angle, and the other two angles are equal to each other. Find the measure of all three angles and sketch the triangle.

Example. Find all angles in the following diagram:



Another problem solving technique is using **similar triangles**. Define similar triangles, draw a few examples.

Example. Mr. Schachner is standing a quarter mile away from a lighthouse. At his towering height of 5 feet 6 inches, he casts a 40-foot-long shadow. About how tall is the lighthouse?

Example. The Americans with Disabilities Act requires that an accessible ramp which rises 9 feet must be at least 108 feet long. How long must a ramp be to rise 25 feet?

The Pythagorean theorem

Some right-triangle specific definitions: **leg, hypotenuse.**
State the theorem, make sure to emphasize that it only works for right triangles.

Example. Which is farther away, "as the crow flies": 6 blocks south and 8 blocks east, or 5 blocks south and 12 blocks east?

Example. TV sets are usually advertised by their diagonal screen size, but this can be deceiving! Suppose there are two 65-inch-diagonal TVs, an ultrawide measuring 60 inches wide and a standard measuring 56 inches wide. Find the total area of screen space in each TV.

Part 1 problems

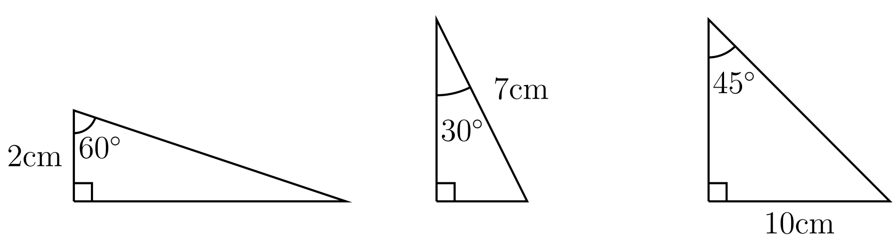
1. Can a triangle have more than one obtuse angle? Explain your reasoning.
2. How can you use a flashlight, a yardstick, and the method of similar triangles to determine the height of a building? Explain your reasoning.
3. An 80-foot-tall radio mast is supported by two cables attached to its top and fixed to stakes in the ground on opposite sides of the mast. If the cables are each 120 feet long, how far apart are the stakes from each other?

Right triangle trigonometry

Explain using similar triangles that if we fix one of the angles of a right triangle, then the shape of the triangle (i.e., the ratios of its sides) stays the same.
These ratios have names: define **sine, cosine, tangent**. We can use the Pythagorean theorem to build a table of values:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$45°$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$60°$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$

Example. For each of the following triangles, determine all sides and angles.



Example. A surveyor wants to measure the width of a large river. After marking one point at the opposite side of the river and walking sixty feet along the river, she measures an angle of 60°between the marked point and where she started. How wide is the river?

Example. An airplane takes off with an angle of 15 degrees at a speed of 180mph. After 30 seconds, what is its elevation? (You can use the approximation $\sin 15° \approx 1/4$.)

Part 2 problems

1. For each of the following descriptions, draw the triangle with labeled sides and angles. Then determine the remaining sides and angles.
   (a) A right triangle with one angle equal to 30°. The length of the side between that angle and the right angle is $6cm$.
   (b) An isoceles right triangle whose hypotenuse is 1 mile long.
   (c) An obtuse isoceles triangle with base 20 feet and height 5 feet.
2. Mt. Everest is about 29,000 feet tall. How far away from it do I have to stand for my viewing angle of it to be a comfortable 60°?
3. A *parsec* is an absurdly long unit of distance used in cosmology and *Star Wars*. It is defined in the following way: if I'm standing 1 parsec away looking at Earth and the Sun, then the difference in viewing angle is one *arcsecond*, which is a tiny angle equal to 1/3600 of a degree. Given that the tangent of an arcsecond is approximately 1/200000, and the Sun is roughly 93 million miles from Earth, about how long is a parsec?

Day 11

Probability I: Counting, Permutations & Combinations

March 31, 2026

Today: Probability, day 1.
Textbook sections 11.1 thru 11.5.

The Fundamental counting principle

How do we count a big collection of things without listing them all out and counting them one by one?

Example. Mr. Schachner is making his favorite meal: grilled cheese with soup.

- He can either use white bread or sourdough.
- He can make it with American cheese, Havarti, or cheddar.
- He can have tomato soup or french onion soup.

How many different meal choices does he have in total?

Make it to the **Fundamental counting principle**: When making a number of independent choices, the total number of choices is the individual numbers of options, all multiplied together.

Example. In a standard game of Mastermind, the secret code is a sequence of 4 pegs, each taken from a choice of 6 colors (duplicates are allowed). How many possible secret codes are there?

Permutations and combinations

Sometimes we want to count a large collection of objects where the choices we have are not independent:

Example. Three sprinters— Allyson Felix, Bob Hayes, and Carl Lewis— are competing in a 100m dash exhibition race. How many different ways can the three of them finish, assuming there are no ties?

Use multiplication to show that the number of ways to order an n-element set is the product $1 \times 2 \times 3 \times \cdots \times n$. Define **factorial.**

Example. What is $5!$? What is $5!/3!$? What is $127!/126!$?

Sometimes we don't need to order all of the elements of a set, only some of them:

Example. Every month, Mr. Schachner chooses four movies to show at his weekly movie nights that month. This month, he had a list of six movies he could choose from. How many ways can he choose a schedule for the month?

Arrive at the formula for permutations: $nPr = \dfrac{n!}{(n-r)!}$.

What if order doesn't matter?

Example. Ms. Beaupre is at the plant store and finds 9 different plants she likes, but she can only take 3 home. How many combinations of plants can she buy?

Define **permutations** vs. **combinations**.
Write the formula for combinations: $nCr = \dfrac{n!}{r!(n-r)!}$.

Part 1 problems

1. A standard US license plate has three letters, followed by four digits (like ABC-1234). How many possible license plates are there?
2. Mr. Schachner tells Ms. Beaupre that his computer password is super secure: it's just a string of 8 random lowercase letters, which no one could ever guess. What Mr. Schachner doesn't know is that Ms. Beaupre is a skilled hacker, and her computer can check 8 billion passwords every second! About how long will it take her to crack his password? (You can use that 8 billion is about 26^7.)
3. A standard Personal Identification Number (PIN) consists of 4 digits. How many possible PINs are there? How many PINs are there whose digits are all different?
4. Write an expression for the number of possible 5-card hands which can be dealt from a standard 52-card deck.

Intro to probability

Some probability questions to build up to a pattern:

Example. What's the probability of...

1. flipping heads on a fair 2-sided coin?
2. rolling a 6 on a standard 6-sided die?
3. rolling an even number on a standard die?
4. flipping three heads in a row on a fair 2-sided coin?

Arrive at the general formula

$$\text{Probability of X} = \frac{\text{number of ways to get X}}{\text{total number of possibilities}}.$$

This shows how our earlier efforts to count things faster are massively helpful in probability.

Examples to finish things out:

Example. Using the answer to question 4 above, write an expression for the probability of dealing a royal flush from a standard 52-card deck.

Example. What's the probability of getting the same number of heads and tails when flipping 4 coins? What about with 5 coins?

Example. No one in history has ever created a perfect March Madness bracket. Mr. Schachner knows nothing about college basketball, so always just fills his out randomly. If there are 64 teams, what are the chances of him picking a perfect bracket?

Example. What are the chances that two randomly selected people have the same birthday?

Day 12

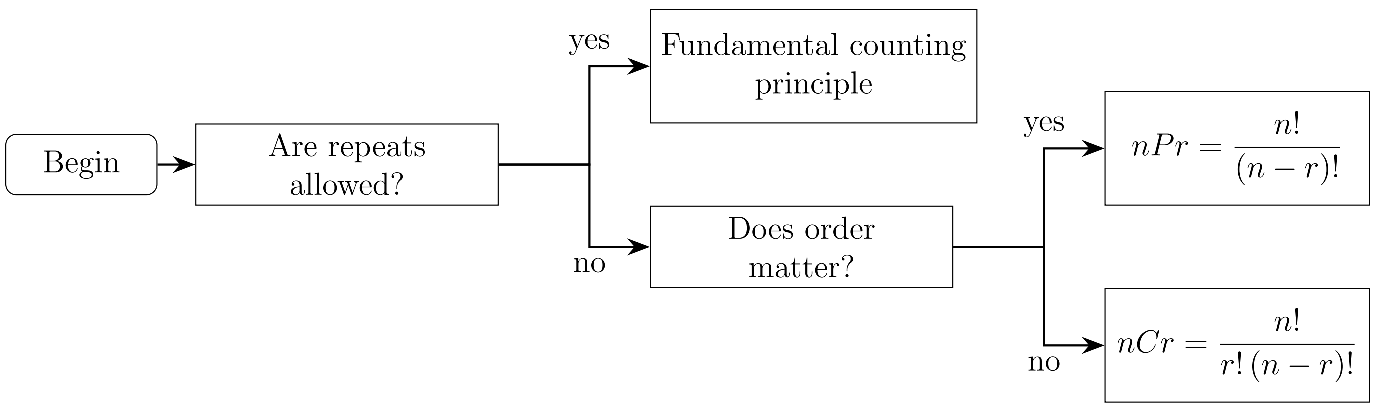
Probability II: Rules, Conditional Probability & Expected Value

April 7, 2026

Today: Probability, day 2.
Textbook sections 11.6 thru 11.8.

Review from last time

Counting flowchart:



The probability of an event is the number of ways the event can happen, divided by the total number of possible outcomes.

Probability with connectives

Recall *connectives*: "not", "or", "and".

The rule for "not"

Example. Suppose I roll two dice. What is the probability that I will **not** roll two ones?

Use this example to motivate the probability rule for "not": **the probability of an event not occurring is 1 minus the probability of the event occurring.**

The rule for "or"

We need the following definition:

Definition. Two events are **mutually exclusive** if it is impossible for both to happen at the same time.

Give some examples of mutually exclusive and non-mutually exclusive events.

Example. What is the probability of drawing a spade or a heart from a deck of cards?

Arrive at the rule for "or" for mutually exclusive events: **If A and B are mutually exclusive, then the probability of A or B is the probability of A plus the probability of B.**

What about for non-mutually exclusive events?

Example. What is the probability of drawing a spade or a queen from a deck of cards?

Show that we can count using inclusion-exclusion to arrive at the rule

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B).$$

The rule for "and"

Again, we need a definition:

Definition. Two events are *independent* if one happening has no influence on the probability of the other happening.

Talk through some examples of independent and dependent events.

Example. Ms. Beaupre rolls two dice. What is the probability that the first die comes up even and the second comes up odd?

Arrive at the rule for "and" for independent events: **If two events are independent, then the probability of both happening is given by multiplying together their individual probabilities.**

What about for dependent events?

Example. Suppose we have a deck with three red cards and two black cards. What is the probability of drawing a black card and then drawing a red card (without replacing the first black card)?

Probabilities with "at least once"

Example. Suppose I flip four coins. What is the probability that I flip no heads? What is the probability that I flip at least one head?

Arrive at the rule for "at least once": **The probability of an event happening at least once is 1 minus the probability of the event not happening at all.** It is usually much easier to find the probability of something not happening at all.

Part 1 Problems

1. Of the standard poker hands (high card, pair, two pair, three-of-a-kind, straight, flush, full house, four-of-a-kind, straight flush, royal flush), find two which are mutually exclusive.
2. On any given day, there's a 10% chance that Mr. Schachner oversleeps his alarm. Find the probability that he oversleeps at least once in a five-day work week.
3. A college baseball team has five seniors, three juniors, and one sophomore. The batting lineup is decided randomly. Find the probability that the first three hitters are two juniors and a sophomore, in that order.

Conditional probability

Before, we saw with dependent events, that

$$P(A \wedge B) = P(A) \times P(B \text{ given } A).$$

That second probability is called a **conditional probability**. We write $P(B \mid A)$.

Conditional probability has a nice formula:

$$P(B \mid A) = \frac{\text{Number of ways to get } A \text{ and } B}{\text{Number of ways to get } A.}$$

Example. What is the probability that a random day is a Tuesday, given that it is a weekday?

Example. What is the probability that a randomly selected two-digit number is divisible by 4, given that its last digit is 2?

Expected value

Example. Suppose we play the following game. I roll a die, and if it comes up as a 6, I pay you \$6. How much would you pay to play this game?

Expected value is the mathematical answer to these sorts of questions: if an event produces outcomes with a range of values, the **expected value** is the average value produced.

The rule for expected value is: **the expected value is given by multiplying each outcome by its probability and adding the results together.**

Example. What's the expected value of rolling a 6-sided die? What about a 12-sided die? What about an N-sided die?

Part 2 Problems

1. Suppose A and B are events such that $P(A) = 1/2, P(B) = 1/3$. If A and B are independent, find $P(A \wedge B), P(A \mid B)$, and $P(B \mid A)$.
2. Repeat the previous problem, this time under the assumption that A and B are mutually exclusive events.
3. Use expected value to write an argument for or against playing the lottery. Feel free to make up your own (realistic) numbers to support your conclusion.

Day 13

Statistics I: Data & Measures of Central Tendency

April 14, 2026

Today: Statistics, day 1.
Textbook sections 12.1,12.2,12.3.

Intro to statistics

Statistics is all about making sense of data. Some useful definitions:

- **Population**: the set of individuals we want to understand using our data.
- **Sample**: A subset of the population we collect data from.

A sample is **random** if no individual in the population is more likely to be chosen than any other. Random samples are important because if a sample is not random, the data will be less useful:

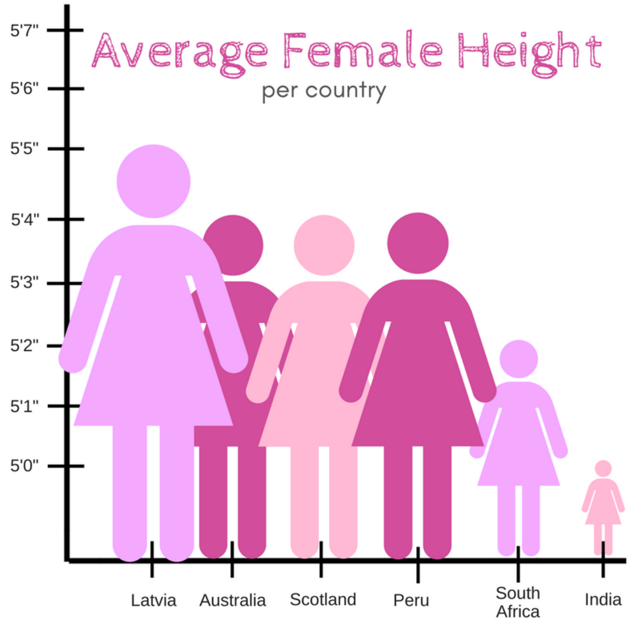
Example. A New York-based car manufacturer wants to know what proportion of Americans get to work by car. They set up a few booths at various street corners in Manhattan, and ask passersby how they get to work. After collecting roughly a thousand responses, they report that only 10% of Americans get to work by car, wildly undershooting the more common figure of 80-90% reported by other samples. What's wrong with their methods?

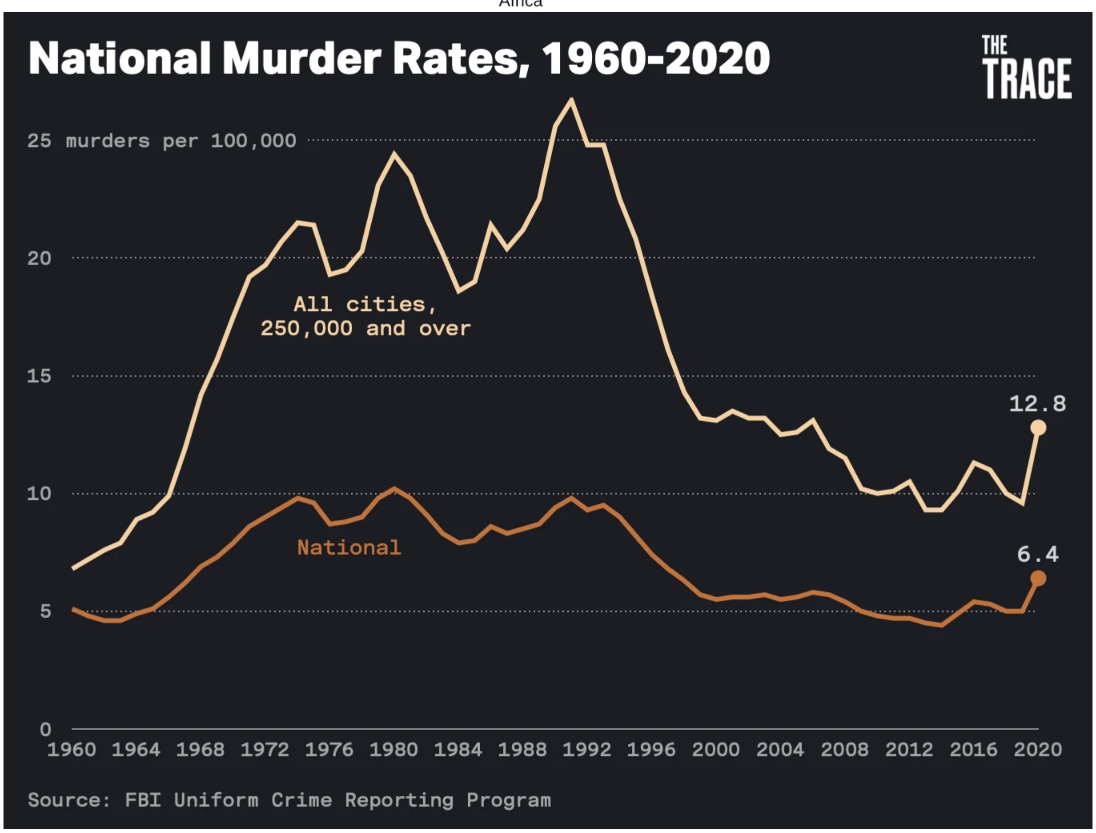
Once data is collected, the next question becomes: how can we organize and represent it visually? Talk about frequency distributions, histograms, frequency polygons.

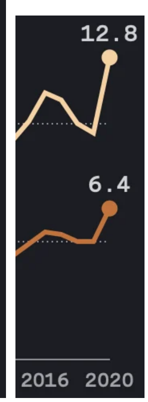
Example. The high temperatures, in degrees Fahrenheit, for a 2-week period are: 52, 48, 45, 51, 58, 62, 65, 61, 54, 50, 53, 59, 64, 67. Organize this data using a frequency distribution, histogram, and frequency polygon.

Sometimes, however, data representations can be misleading, intentionally or not:

Example.







Measures of central tendency

Once our data is organized, we want to analyze it. The most common way to do this is by calculating what a "typical" data point looks like. These are **measures of central tendency**.

Three classical ones:

- Mean: Average of all values.
- Median: Middle data point.
- Mode: Most common data point.

Example. Examples of data sets where different measures of central tendency are better / worse:

1. An exam given to 12 students where the grades are

$$47, 50, 60, 70, 75, 97, 97, 100, 100, 100, 100, 100$$

   (Mean: 83, median: 97, mode: 100.)

2. Household income in the United States. Mean: 120k/year. Median: 75k/year.

Part 1 Problems

1. After asking a sample of adults what the length of their commutes are, a study finds that the median is much less than the mean. What conclusion can we draw from this observation? Does it make sense?
2. A common statistics joke goes, "the average human has less than two arms and half a uterus." Explain what this means in terms of measures of central tendency.
3. In order to determine people's grocery shopping habits, Mr. Schachner stands outside a Whole Foods grocery store and asks 100 arbitrary people how often they go grocery shopping. Is this a random sample? Why or why not?

Measures of dispersion

Another way we can analyze data is by measuring how "spread out" it is. Two samples can have the same measures of central tendency, but tell very different stories:

Example. Despite being known as one of the hottest places on earth, the Sahara desert has an average temperature of only 86 degrees Fahrenheit. This is roughly the same as the average temperature in the Maldives, an island country in the Indian Ocean. But most people would much rather live in the climate of the Maldives. Why?

The simplest measurement of dispersion is the **range**: top data point minus bottom data point. Easy to calculate but not always useful:

Example. Explain why range is a poor measure of dispersion when sampling ages within a population.

The **standard deviation** is more complicated but takes all data points into account:

- For each data point, find the squared distance from the mean.
- Add up all the squared distances and divide by $n-1$, where n is the number of data points.

Example. Find the mean and standard deviation of the following data sets:

$$\{4000, 4000, 5000, 6000, 6000\}, \{1, 3, 5, 5, 6\}.$$

Part 2 Problems

1. Which would you expect to have a higher mean temperature, upstate New York or the San Francisco Bay area? Which would you expect to have a higher standard deviation?
2. A surprising fact about the Solar System is that over 99% of its total mass is contained within the Sun alone. What does this tell you about the mean, median, and standard deviation of the sizes of objects in the Solar System? (For example, which is larger, the mean or median?)

Day 14

Statistics II: Normal Distribution & Correlation

April 21, 2026

Today: Statistics, day 2.
Textbook sections 12.4,12.5,12.6.

Review from last time

Statistics is a toolbox for analyzing data.

Two ways we can analyze data:

- **Central tendency**: mean, median, mode.
- **Dispersion**: range, standard deviation.

The normal distribution

Define the normal distribution (aka bell curve): a distribution that many data sets throughout nature follow. Draw a picture.

Two numbers are enough to describe any normal distribution: the mean and standard deviation. Define the **68-95-99.7 rule**.

Example. The weight of the ruby-throated hummingbird is on average 3 grams, with a standard deviation of 0.3 grams. If the weights within a population are normally distributed, find a weight range into which 95% of ruby-throated hummingbirds fall.

Example. On an exam with maximum score 100%, students average a score of 86%, with a standard deviation of 7%. What proportion of the students received a score of at least 79%?

We can assign scores based on a data point's position within the distribution:

- **z-score**: number of standard deviations above or below the mean.
- **percentile**: percentage of data points below the data point.

Example. A sample of 20 adult humans were asked to measure the width of their hair in micrometers. The following data set was produced:

$$\{83, 71, 76, 92, 80, 65, 87, 79, 75, 100, 84, 68, 95, 77, 81, 60, 89, 73, 85, 80\}$$

This data is approximately normally distributed, with mean 80 micrometers and standard deviation 10 micrometers. Find the z-score and percentile of the individual whose hair is 65 micrometers thick.

Part 1 Problems

1. Anthony, Bella, and Chuck took a test, and the test grades were normally distributed. Anthony scored one standard deviation below the mean, Bella scored exactly at the mean, and Chuck scored one standard deviation above the mean. Find their respective percentiles.
2. Using the above data set for hair width, find the width of hair corresponding to a z-score of 2. What percentile would such an individual be?
3. Suppose a data set is **not** normally distributed. Can we still calculate the percentile corresponding to the mean? What about the median?

Scatterplots, correlation, and regression lines

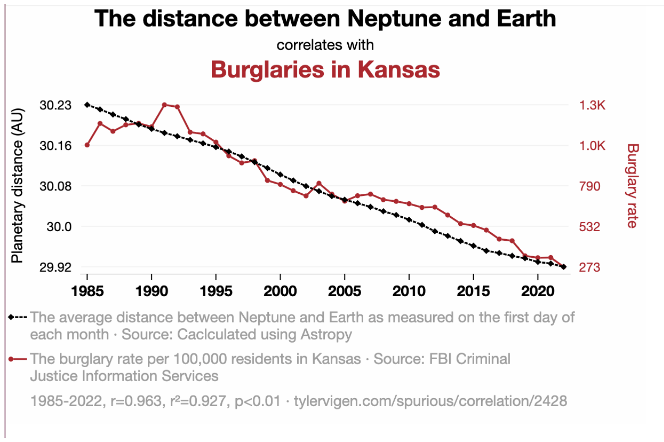
Many questions in statistics take the form "how does variable X relate to variable Y?". For example:

- Are stronger swimmers taller on average?
- Do people with larger social networks earn more money on average?
- Do people who eat fish more often have lower rates of heart disease?

Studies which try to answer these questions sample a large group of people and measure both variables. This gives a large number of pairs (variable X, variable Y) which we can then plot in a 2D plane. This is a **scatterplot**. Draw an example.

Scatterplots are good for answering questions of **correlation**: how strong the relationship is between two variables. However, **correlation is not causation!**

Example. Below is a chart of the distance between the planet Neptune and Earth, along with data on the number of burglaries in Kansas. Does this mean that people commit fewer burglaries because Neptune is closer?



Different kinds of correlation:

- Positive correlation: as variable X increases, variable Y tends to increase too.
- Negative correlation: as variable X increases, variable Y tends to decrease.
- No correlation: variables X and Y have no relationship.

Sometimes a correlation is so strong that you can draw a line which passes close to– or through– most of the data points. This is called a **regression line**, or sometimes **line of best fit**. There is a formula for the line of best fit, but we usually use calculators to find it.

Part 2 Problems

1. For each of the following, say whether you expect a positive, negative, or no correlation.
   (a) Household income and number of cars owned;
   (b) Elevation above sea level and average temperature;
   (c) Distance from the Equator and average temperature;
   (d) Distance from the Prime Meridian and average temperature.

Day 15

Final Exam Review

April 28, 2026

Today: Final exam review.

Algebra practice

Example. A dress shirt and suit jacket together cost \$165. The same dress shirt and a pair of pants together cost \$125. The suit jacket and pants together cost \$180. How much do all three cost together?

Example. I'm thinking of two numbers whose sum is 55 and whose difference is 19. What is their product?

Example. When Mr. Schachner was in the fifth grade, his parents were four times his age. When he graduated high school six years later, his parents were three times his age. How old were his parents when he was born?

Venn diagram problems

Example. A survey of 2000 New Yorkers asked whether respondents supported the Giants and / or the Jets. 1450 respondents supported the Giants, 280 supported neither, and 400 supported both. How many supported the Jets but not the Giants?

Example. A special election contains two ballot measures, Measure A and Measure B; for a ballot to be counted, a voter must vote either Yes or No to both measures. Measure A passed with 53% of voters voting yes, while Measure B failed with only 35% of voters voting yes. The election board observed that only 8% of voters voted yes to both. What proportion of the voters voted no to both?

Truth tables

Example. Write each sentence in symbolic form, then construct a truth table for the sentence.

(a) Either the wallpaper goes or I do.

(b) If I had a nickel for every time I taught MATH112, I'd have two nickels.

(c) You can't have your cake and eat it too.

Arguments

Example. Determine whether each argument is valid using a truth table.

(a)
$$\begin{array}{l} p \wedge q \\ q \to c \\ \hline \sim c \end{array}$$

(b)
$$\begin{array}{l} \sim(\sim x \wedge \sim y) \\ \sim y \\ \hline x \end{array}$$

(c) Every square has three sides
Mr. Schachner's bedroom is a square
———
Mr. Schachner's bedroom has three sides.

Converting between bases

Example. Write 25 in base 2, base 5, base 12, and base 16.

Example. Convert the following to base 10: $(10010110)_2, (A05)_{11}, (313)_4, (313)_5$.

Base arithmetic

Example. Complete the following arithmetic, leaving your answer in the same base as the problem:

$$\begin{array}{r} (73)_8 \\ \underline{+(5)_8} \end{array} \qquad \begin{array}{r} (B00)_{14} \\ \underline{-(10)_{14}} \end{array} \qquad \begin{array}{r} (111)_2 \\ \underline{\times(11)_2} \end{array} \qquad \begin{array}{r} (99)_{11} \\ \underline{+(1)_{11}} \end{array} \qquad \begin{array}{r} (32)_9 \\ \underline{\times(32)_9} \end{array}$$

Volume and surface area

Example. Ms. Beaupre's favorite bowling ball is 9 inches in diameter. It has three finger holes, each of which was made by drilling a cone-shaped hole of diameter 0.6 inches. Find the volume of the bowling ball, leaving your answer in terms of pi.

Example. A *Quonset hut* is a common architectural style which is approximately half a cylinder, as shown below.

Assume the height of the hut is 2 meters, the door is 1 meter by 1.5 meters, and the two windows are squares 15cm on a side. (The back side is bare.) Find the volume of the hut. If a gallon of paint can coat 35 square meters, how much paint is required to paint the whole hut?

Right triangle trigonometry

Example. A triangle has two angles measuring 39 degrees and 90 degrees; the side between these two angles has length 18 feet. Find its area and perimeter. You may use that $\sin 39° \approx 17/27, \cos 39° \approx 7/9$, and $\tan 39° \approx 17/21$.

Example. Find the height of an equilateral triangle whose sides measure 6cm.

Example. Mr. Schachner's 6-inch-tall drinking glasses are wider at the top than the bottom. He measures the bottom to have diameter 2 inches and the top to have diameter 3 inches. How much water can one glass hold?